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(Residential Autonomous College under University of Calcutta)

**B.A./B.Sc. FIRST SEMESTER EXAMINATION, JANUARY 2015**

**FIRST YEAR**

Date : 05/01/2015 **MATHEMATICS (Honours)**

Time : 11 am – 3 pm **Paper : I** Full Marks : 100

## [Use a separate Answer Book for each group]

## Group – A

(***Answer any five questions from Q.No. 1 - 8***) [5×5]

1. a) Let A be a nonempty subset of $\mathbb{R}$. Prove that there does not exist any surjective mapping $f: A \to \mathcal{P}(A)$, where $\mathcal{P}(A)$ is the power set of A. [3]

   b) Show that the number of different reflexive relations on a set of n elements is $2^{n^2-n}$. [2]

2. Find the smallest equivalence relation on $\mathbb{R}$ containing the line $x-y=1$. [5]

3. Prove that every group of order less than 6 is commutative. [5]

4. Let G be a group in which $(ab)^3 = a^3b^3$ for all $a, b \in G$. For $n \in \mathbb{N}$, define $K_n = \{x^n : x \in G\}$.

   Show that : a) $K_n \leq G$ for n = 1,2 [2]

   b) $K_n \leq G$ for all $n \in \{2^i 3^j : i, j \in \mathbb{N}\}$ [3]

5. a) For $p, q \in \mathbb{N}$ show that the smallest subgroup of $\mathbb{Z}$ containing $p\mathbb{Z} \cup q\mathbb{Z}$ is $\ell\mathbb{Z}$, where $\ell = \gcd(p,q)$. [2]

   b) Show that $[\mathbb{Z} : m\mathbb{Z}] = |m|$ for any $m \in \mathbb{Z} \setminus \{0\}$. [3]

6. Let G be a cyclic group of order 144.

   Find : a) the number of subgroups of G,

   b) the number of subgroups of G of order 12

   c) the number of elements with order 12

   d) [G : H], where $H \leq G$ with $|H| = 12$.

   Further, if G be non-commutative, show that G has an element of order 3. [5×1]

7. State Lagrange's Theorem on finite groups. State and prove its converse in case of cyclic groups. [1+4]

8. a) Prove that every element in the alternating group $(A_n, \cdot)$ is a product of 3-cycles. [3]

   b) Find the order of $(1\ 2\ 3\ 4) \cdot (5\ 6\ 7)$ in $S_7$. [2]

(***Answer any five questions from Q.No. 9 - 16***) [5×5]

9. a) Let S be a bounded subset of $\mathbb{R}$ with sup S = M and inf S = m. Prove that the set $T = \{x - y : x \in S, y \in S\}$ is a bounded set and sup T = M – m. [3]

   b) Let S be a nonempty subset of $\mathbb{R}$ which is bounded below and a = inf S. If $a \notin S$, prove that a is a limit point of S. [2]

10. a) A real number $\alpha$ is said to be algebraic if there are integers $a_0, a_1, \ldots, a_n$, not all zero, such that $a_0 + a_1\alpha + \ldots + a_n\alpha^n = 0$. Prove that the set of all algebraic numbers in $\mathbb{R}$ is countable. [3]

    b) Show that the set of all circles in the plane $\mathbb{R}^2$ having rational radii and centres with rational co-ordinates is countable. [2]

11. a) Prove or disprove : "If A, B are open sets in $\mathbb{R}$, then so is $A + B = \{a + b : a \in A, b \in B\}$. [3]

    b) Find all $A, B \subseteq \mathbb{R}$ satisfying $A^0 = \overline{B}$. [2]

12. Show that the sequence $\{u_n\}_{n\in\mathbb{N}}$ defined by $0 < u_1 < u_2$ and $u_{n+2} = \sqrt{u_{n+1}u_n}$ for $n \geq 1$, converges to $\sqrt[3]{u_1 u_2^2}$. [5]

13. State and prove Bolzano-Weirstrass' Theorem on the existence of convergent subsequences of a bounded sequence in $\mathbb{R}$.
Is the theorem valid if we drop the assumption on boundedness of the sequence? [4+1]

14. a) If $\{x_n\}$ be a null sequence, prove that $e^{x_n} \to 1$. [3]
b) Define a Cauchy sequence in $\mathbb{R}$. Give an example to show that a real sequence $\{x_n\}$ need not be Cauchy even if $|x_n - x_{n+1}| \to 0$ as $n \to \infty$. [1+1]

15. a) Let $S \subseteq \mathbb{R}$ be a bounded set. Describe the smallest closed interval containing S. [3]
b) Give an example of a sequence with countably infinite number of subsequential limits. [2]

16. a) Prove that $\lim\limits_{x\to 0} \cos\frac{1}{x}$ does not exist. [2]
b) Let I = (a,b) be a bounded open interval and $f : I \to \mathbb{R}$ be monotone decreasing function on I. If f is bounded above on I, then prove that $\lim\limits_{x\to a^+} f(x) = \sup\limits_{x\in(a,b)} f(x)$. [3]

## Group – B

**(*Answer <u>any three</u> questions from <u>Q.No. 17 - 21</u>*)** [3×5]

17. A triangle has the lines $ax^2 + 2hxy + by^2 = 0$, for two of its sides and the point (p,q) for its orthocentre. Show that the equation of its third side is $(a+b)(px+qy) = aq^2 + bp^2 - 2hpq$. [5]

18. Lines are drawn through the foci of an ellipse perpendicular respectively to a pair of conjugate diameters and intersect at Q. Show that the locus of Q is a concentric ellipse. [5]

19. If g be a variable tangent to the conic $\frac{\ell}{r} = 1 - e\cos\theta$, show that the locus of the foot of the perpendicular from the pole on g is the circle $(e^2-1)r^2 + 2\ell e r\cos\theta + \ell^2 = 0$. [5]

20. Reduce the equation $3(x^2+y^2) + 2xy = 4\sqrt{2}(x+y)$ to its canonical form. Name the conic and find the eccentricity of the conic. [3+1+1]

21. Find the locus of the poles of the chords of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which subtend right angles at the centre of the ellipse. [5]

**(*Answer <u>any two</u> questions from <u>Q.No. 22 - 24</u>*)** [2×5]

22. ABCD is a parallelogram and E is the mid-point of CD and F is a point on AE such that $AF = \frac{2}{3}AE$.
Show, by vector method, that F lies on the diagonal BD and $BF = \frac{2}{3}BD$. [4+1]

23. a) A force of magnitude 21 unit along $2\hat{i} + \hat{j} - 2\hat{k}$ is acted on a particle at A(1,3,6) and displaces it at B(3,6,9). Find the work done. [2]
b) Show that an arbitrary vector $\vec{v}$ can be expressed in terms of any three non-coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ according to $\vec{v} = \frac{[\vec{v},\vec{b},\vec{c}]}{[\vec{a},\vec{b},\vec{c}]}\vec{a} + \frac{[\vec{v},\vec{c},\vec{a}]}{[\vec{a},\vec{b},\vec{c}]}\vec{b} + \frac{[\vec{v},\vec{a},\vec{b}]}{[\vec{a},\vec{b},\vec{c}]}\vec{c}$. [3]

24. Find the shortest distance between two lines, one of them passes through A(6,2,2) and parallel to the vector $\hat{i}-2\hat{j}+2\hat{k}$ and the other through C(–4,0,–1) and parallel to the vector $3\hat{i}-2\hat{j}-2\hat{k}$. Find the points where the lines meet the shortest distance line. [4+1]

**(*Answer* _*any five*_ *questions from* _*Q.No. 25 - 32*_)** [5×5]

25. Show that the general solution of the equation $\frac{dy}{dx}+Py=Q$ can be written in the form $y=K(u-v)+v$, where K is a constant and u & v are its two particular solutions. [5]

26. Find an integrating factor of the differential equation $(y^2+2x^2y)dx+(2x^3-xy)dy=0$ and hence solve it. [5]

27. Using the transformation $u=x+y, v=x^2+y^2$, reduce the following differential equation into Clairaut's form and hence solve it : $(x^2+y^2)(1+p)^2-2(x+y)(1+p)(x+yp)+(x+yp)^2=0$, $p=\frac{dy}{dx}$. [5]

28. Solve by the method of variation of parameters: $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=\frac{e^{-x}}{x^2}$. [5]

29. Solve by the method of undetermined coefficients the differential equation $(D^2-3D)y=x+e^x\sin x, \left(D\equiv\frac{d}{dx}\right)$. [5]

30. Solve: $(1+x+x^2)\frac{d^3y}{dx^3}+(3+6x)\frac{d^2y}{dx^2}+6\frac{dy}{dx}=0$. [5]

31. Find the family of curves cutting the family of parabolas $y^2=4ax$ at 45°. [5]

32. Solve: $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\log(1+x)$ [5]

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